

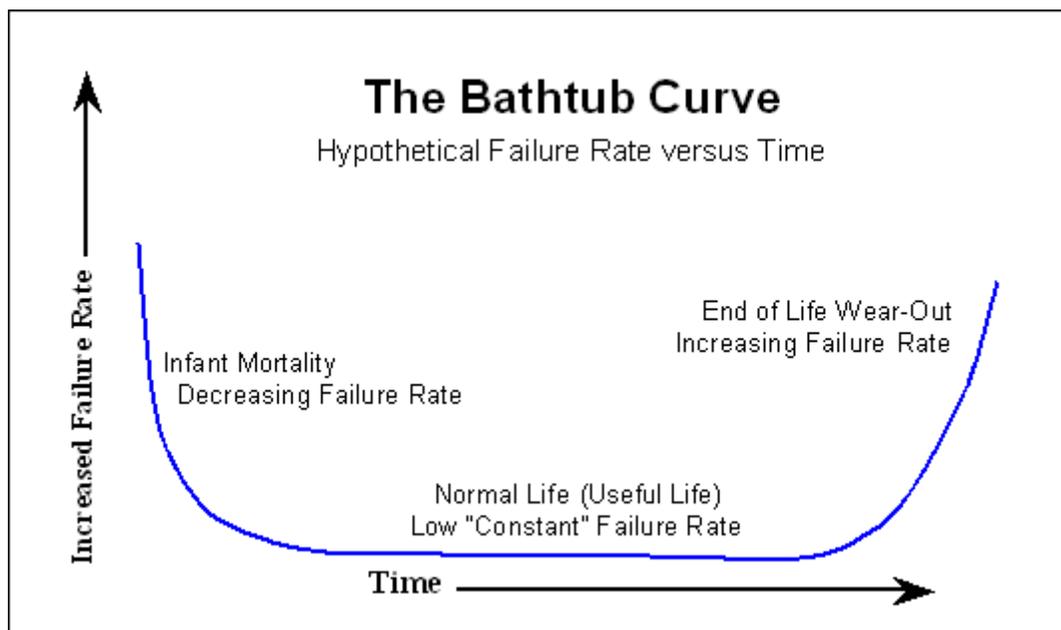
## **Using the Weibull Method for Determining Environmental Stress Screening Duration**

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## 1.0 Introduction

The purpose of using statistical distributions is to make inferences about some behavior of a population by using a sample of the population. Since the characteristics (probability density) of the data is not always the same, many different statistical distributions exist. Some of these statistical distributions include Normal, Lognormal, Chi-square, F, Exponential, Gamma and Weibull. For screening electronic assemblies, the distribution of interest is the failure density over time. A model for failure density (shown below) is often called the bath-tub curve.

This document will describe the procedural steps required to determine the parameters for a Weibull distribution and then use the Weibull distribution to choose the minimum exposure time for an environmental stress screening process. In this paper, sample data will be used to demonstrate each of the mathematical expressions and create plots which will be ultimately used to determine screening duration.



The bath-tub curve is composed of three distinct regions: the decreasing hazard rate region (infant mortality), constant hazard rate region (useful life) and the increasing hazard rate region (wear out). The objective of screening electronic assemblies is to accelerate the aging of the assemblies so that they have reached the useful life region before shipment to the customer. The most widely used mathematical model for describing the failure behavior of electronic assemblies over time is the Weibull distribution function. This document will describe the procedural steps required to determine the parameters for a Weibull distribution and then use the Weibull distribution to choose the minimum exposure time for an environmental stress screening process. In this paper, sample data will be used to demonstrate each of the mathematical expressions and create plots which will be used to determine screening duration.

## 2.0 The Weibull Function

The Weibull three parameter probability density function:

$$f(t) := \frac{\beta}{\eta - t_0} \cdot \left[ \left( \frac{t - t_0}{\eta - t_0} \right)^{(\beta - 1)} \cdot \exp \left[ - \left( \frac{t - t_0}{\eta - t_0} \right)^\beta \right] \right] \quad (1)$$

The Weibull three parameter hazard rate function:

$$h(t) := \frac{\beta}{\eta - t_0} \cdot \left( \frac{t - t_0}{\eta - t_0} \right)^{(\beta - 1)} \quad (2)$$

The Weibull three parameter reliability function:

$$R(t) := \exp \left[ - \left( \frac{t - t_0}{\eta - t_0} \right)^\beta \right] \quad (3)$$

where:

- $\beta$  = the shape parameter
- $\eta$  = the characteristic life
- $t_0$  = the minimum lifetime
- $t$  = time

These functions are the three parameter Weibull functions. When applied to the infant mortality region of the bath-tub curve, electronic assemblies generally do not have a minimum lifetime ( $t_0 = 0$ ). A situation where a product may have a minimum lifetime occurs when a product needs to fully execute a loop of a test before the first opportunity for failure reporting takes place. When the minimum lifetime parameter is removed from the Weibull functions, a set of two parameter Weibull functions result.

The Weibull two parameter probability density function:

$$f(t) := \frac{\beta}{\eta} \cdot \left[ \left( \frac{t}{\eta} \right)^{(\beta - 1)} \cdot \exp \left[ - \left( \frac{t}{\eta} \right)^\beta \right] \right] \quad (4)$$

The Weibull two parameter hazard rate function:

$$h(t) := \frac{\beta}{\eta} \cdot \left(\frac{t}{\eta}\right)^{(\beta - 1)} \quad (5)$$

The Weibull two parameter reliability function:

$$R(t) := \exp\left[-\left(\frac{t}{\eta}\right)^\beta\right] \quad (6)$$

Since  $R(t) = 1 - F(t)$ , it can be substituted into equation 6

$$1 - F(t) := \exp\left[-\left(\frac{t}{\eta}\right)^\beta\right] \quad (7)$$

The shape of the Weibull hazard rate curve is determined by the two constants  $\beta$  (the shape parameter) and  $\eta$  (the characteristic life). When  $\beta < 1$ , the hazard rate curve shows a decreasing hazard rate which corresponds to the infant mortality region of the bath-tub curve. When  $\beta = 1$ , the function reduces to the exponential distribution and the hazard rate curve shows a constant hazard rate which corresponds to the useful life region of the bath-tub curve. When  $\beta > 1$ , the hazard rate curve shows an increasing hazard rate which corresponds to the wear-out region of the bath-tub curve.

The values for  $\beta$  and  $\eta$  are determined through an analysis of the results of screening operation. The screening operation will have units which pass, fail or were suspended. A suspended unit is defined as a non-failed unit which has had its screening time terminated before other members of the non-failing population. If all units in the population have their screen started and terminated at the same time and no units are removed before the end of the screen, no suspended units will exist. When a screening operation resembles a first-in first-out queue, units in the population will have slightly different screening durations in which case suspensions are possible.

The data extracted from the screening process includes product part number, sample size, pass/fail and elapsed time (or time of failure). A unit which passes and has a lower elapsed time than a failed unit will be considered a suspended unit. If the unit is executing a test sequence, the number of loops may be extracted as well to ensure that at least one successful pass was executed. A unit which fails to complete a single pass may have been defective prior to the start of the screen. Such units are likely to belong to a different failure distribution. Since each product type may have different failure behavior, the required screen duration (or screen type) is expected to be different for each product type. The product part number extracted from the screening operation is used to distinguish and group the data for the Weibull analysis.

During the Weibull analysis, the results of the screen will be transformed to ensure they fit a Weibull distribution. If successful, the values for  $\beta$  and  $\eta$  will be found and substituted into the Weibull hazard rate function to create a mathematical expression which describes the failure behavior of the units screened. The screening duration can be determined by finding the time for a specific slope or magnitude of the hazard rate curve or through the application of Bayesian Analysis. After the screening time has been found, the 90% confidence limits calculated during the analysis can provide the range of prediction certainty for the resulting screening time. The more failing units in the analysis, the greater the prediction certainty.

### 3.0 References

- 3.1 Jensen, F. and Petersen, N., '*Burn In*', John Wiley and Sons, New York, 1982
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- 3.3 Walpole, R. and Myers, R., '*Probability and Statistics for Engineers and Scientists*', MacMillan Publishing Company, New York, 1985
- 3.4 Kapur, K. and Lamberson, L., '*Reliability in Engineering Design*', John Wiley and Sons, New York, 1977
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- 3.6 Cheney, W. and Kincaid, D., '*Numerical Mathematics and Computing*', Brooks/Cole Publishing Company, Pacific Grove, 1985
- 3.7 Lindgren, T., '*Optimizing ESS Effectiveness using Weibull Techniques*', Proceedings from the Institute of Environmental Sciences, 1986

## 4.0 Procedure

4.1 Data must be gathered from screened units to perform the Weibull fit. Ideally, the number of failed items in the sample should exceed 31 to allow for the greatest accuracy in the determination of the Weibull parameters  $\beta$  and  $\eta$ . A smaller number of failures in the sample can be used and the confidence limits will be adjusted to reflect the loss of prediction accuracy.

4.2 The data extracted from the sample include:

- a. fail (F), pass(P) or suspended (S)
- b. time of failure or suspension
- c. number of test loops completed
- d. sample size

Data points for which the failing unit did not complete a single pass will be excluded from consideration for calculating screening duration since their failure distribution is clearly different than the remaining population. These units were likely to have been defective before entering the screening process. The data from the failing units, however, will be used in the calculations for ranking all failing units. This Weibull plot will treat all failures as though they belong to the same distribution (single mode Weibull).

4.3 The failing and suspended data must then be arranged in ascending order of time and then numerically ranked with integers in reverse order. The first integer for reverse ranking is equal to the sample size. Below is the set of sample data used throughout this paper. In this example, the sample size is 200.

**Table 1 - Ranked Data**

Item	RR "Reverse Rank"	Loops	Time (hours)	Fail (F) / Suspension (S)
A	200	0	0	F
B	199	0	0	F
C	198	1	0.2	F
D	197	2	0.8	F
E	196	3	1.0	S
F	195	6	1.3	F
G	194	11	2.1	F
H	193	24	5.8	F
I	192	25	7.0	S
J	191	40	8.9	F
K	190	69	12.7	F

4.4 Rank order numbers (RN) will be assigned only to the items which failed. This ranking will include item(s) that have failed at time = 0 or number of loops = 0. The expression for the rank order increment (RI) is given by:

$$RI_i = \frac{(\text{sample size} + 1) - RN_{i-1}}{1 + RR_i} \quad (8)$$

The rank order number is given by:

$$RN_i = RI_i + RN_{i-1} \quad (9)$$

Equations (1) and (2) can be combined into the following expression:

$$RN_i = \frac{(\text{sample size} + 1) - RN_{i-1}}{1 + RR_i} + RN_{i-1} \quad (10)$$

where:

$i$  = a particular failed item starting from the failed item with the highest reverse rank number. In this example it is item A since it failed and had the highest reverse rank number (200).

$RR_i$  = the reverse rank number for the item whose rank is being calculated

$RN_{i-1}$  = 0 for the first item considered since there is no previous rank order number

sample size = the total number of items in the sample (failed and suspended items)

For items A through D, the increments will be one because there is no suspended data between them. The rank order numbers for items A through D will, therefore, be 1.00, 2.00, 3.00 and 4.00. Since items E and J were suspended, they will not have rank order numbers. Item F will have a rank order number with a fractional component since suspended item(s) were between it and the next earlier failure. For item F, the calculated value rank order will be:

$$\begin{aligned} RN_i &= \frac{(200 + 1) - 4}{1 + 195} + 4 \\ &= 5.0051 \end{aligned}$$

The previous table with all the rank order values in place is provided below:

**Table 2 - Sample Data Table with Rank Order Values Supplied**

Item	RR "Reverse Rank"	Rank Order Number	Loops	Time (hours)	Fail (F) / Suspension (S)
A	200	1.00	0	0	F
B	199	2.00	0	0	F
C	198	3.00	1	0.2	F
D	197	4.00	2	0.8	F
E	196	-	3	1.0	S
F	195	5.0051	6	1.3	F
G	194	6.0102	11	2.1	F
H	193	7.0153	24	5.8	F
I	192	-	25	7.0	S
J	191	8.0256	40	8.9	F
K	190	9.0359	69	12.7	F

4.5 The median rank values for all the failed units are now added to the table. An approximation for the median rank values is given by:

$$MR_i = \frac{RN_i - 0.3}{\text{sample size} + 0.4} \quad (11)$$

where:

$RN_i$  = the rank order number for the item whose median rank is being calculated

sample size = the total number of items in the sample (failed and suspended items)

For item E, the calculated median rank value will be:

$$MR_i = \frac{3.29 - 0.3}{10 + 0.4} = 0.288 \text{ or } 28.8\%$$

The previous table with all the median rank values in place is provided below:

**Table 3 - Sample Data Table with Median Rank Values Supplied**

Item	RR “Reverse Rank”	Rank Order Number	MR “Median Rank”	Loops	Time (hours)	Fail (F) / Suspension (S)
A	200	1.0000	0.0035	0	0	F
B	199	2.0000	0.0085	0	0	F
C	198	3.0000	0.0135	1	0.2	F
D	197	4.0000	0.0185	2	0.8	F
E	196	-	-	3	1.0	S
F	195	5.0051	0.0235	6	1.3	F
G	194	6.0102	0.0285	11	2.1	F
H	193	7.0153	0.0335	24	5.8	F
I	192	-	-	25	7.0	S
J	191	8.0256	0.0386	40	8.9	F
K	190	9.0359	0.0436	69	12.7	F

4.6 For median rank values, a 90% confidence limit band will be constructed. A 90% confidence band will consist of a 5% confidence limit (lower confidence limit) and a 95% confidence limit (upper confidence limit). The expressions below will determine the 5% and 95% confidence limits for rank order numbers which have no fractional components (integers). When the rank order number has a fractional component, the 5% and 95% limits will be determined by linearly interpolating between the closest upper and lower integers. The rank order number of 5.0051, for example, will be interpolated between 5.0000 and 6.0000.

For the 5% Limit the expression is:

$$CL_{5\%} = \frac{j}{n - j + 1} [F_{\alpha}, 2(n-j+1), 2j] + \frac{j}{n - j + 1} \quad (12)$$

For the 95% Limit the expression is:

$$CL_{5\%} = \frac{j}{n - j + 1} * [F_{\alpha}, 2j, 2(n-j+1)] \quad (13)$$

$$1 + \frac{j}{n - j + 1} * [F_{\alpha}, 2j, 2(n-j+1)]$$

Where:

j = rank order number (integer)

n = number of units in the sample

$[F_{\alpha}, v_1, v_2]$  = the value of  $F_{\alpha}$  for which the distribution has an area of  $\alpha$  to the right of  $F_{\alpha}$

where:

$\alpha$  = the area to the right of the F-value to be determined

$v_1$  = the degrees of freedom for the numerator

$v_2$  = the degrees of freedom for the denominator

The F-distribution function is given by:

$$h(f) = \frac{\Gamma[(v_1 + v_2) / 2] * (v_1 / v_2)^{(v_1/2)}}{\Gamma(v_1/2) * \Gamma(v_2/2)} * \frac{f^{v_1/2-1}}{(1 + v_1f / v_2)^{(v_1 + v_2)/2}} \quad (14)$$

where:

f = the independent variable in the function

Γ = the gamma function, which is given by:

$$\Gamma(n) = \int_0^{\infty} x^{n-1} * e^{-x} dx \quad (15)$$

For integer operands, the gamma function reduces to:

$$\Gamma(n) = (n-1)! \quad (16)$$

If we combine equations (7) and (9) we get:

$$h(f) = \frac{[(v_1 + v_2) / 2 - 1]! * (v_1 / v_2)^{(v_1/2)}}{(v_1/2 - 1)! * (v_2/2 - 1)!} * \frac{f^{v_1/2-1}}{(1 + v_1f / v_2)^{(v_1 + v_2)/2}} \quad (17)$$

To determine the area under the F-distribution curve, equation (17) will be integrated between the limits of 0 and  $F_{\alpha}$ .  $F_{\alpha}$  is the value we seek which provides a result of 0.95 for the integral. This integral calculates the area left of  $F_{\alpha}$ . Due to the complexity of the integral, numerical techniques (trapezoid method) will be used to solve the following expression:

$$0.95 = \int_0^{F_{\alpha}} \frac{[(v_1 + v_2) / 2 - 1]! * (v_1 / v_2)^{(v_1/2)}}{(v_1/2 - 1)! * (v_2/2 - 1)!} * \frac{f^{v_1/2-1}}{(1 + v_1f / v_2)^{(v_1 + v_2)/2}} * df \quad (18)$$

For the example data, the 5% and 95% confidence limit values for item B are calculated as follows:

n = 200 (sample size)

j = 2 (rank order number for item B)

For the 5% Limit the, solution is:

$$\begin{aligned}
 & \frac{2}{200 - 2 + 1} \\
 CL_{5\%} = & \frac{2}{[0.05, 2(200-2+1), 2*2] + \frac{2}{200 - 2 + 1}} \\
 & = \frac{0.01005}{5.64 + 0.01005} \\
 & = 0.0018 \text{ or } 0.18\%
 \end{aligned}$$

For the 95% Limit, the solution is:

$$\begin{aligned}
 & \frac{2}{200 - 2 + 1} * [0.05, 2*2, 2(200-2+1)] \\
 CL_{95\%} = & \frac{2}{1 + \frac{2}{9 - 2 + 1} * [0.05, 2*2, 2(200-2+1)]} \\
 & = \frac{0.01005 * 2.40}{1 + 0.01005 * 2.40} \\
 & = 0.0236 \text{ or } 2.36\%
 \end{aligned}$$

For item F (rank order number 5.0051), we will interpolate between rank order numbers 5 and 6. The 5% and 95% confidence limit values for rank order numbers 5 and 6 are given below:

**Table 4 - Interpolation Limits**

	rank order 5	rank order 6
5% value	0.0099	0.0131
95% value	0.0453	0.0519

The expression for a simple linear interpolation between two points is given by:

$$y = y_1 + \frac{(y_2 - y_1)}{(x_2 - x_1)} * (x - x_1) \quad (19)$$

For the 5% confidence limit data above with:

$$(x_1, y_1) = (5, 0.0099) \text{ and}$$

$$(x_2, y_2) = (6, 0.0131),$$

the result is:

$$CL_{5\%} = 0.0099 + \frac{(0.0131 - 0.0099)}{(6 - 5)} * (x - 5)$$

$$CL_{5\%} = 0.0099 + 0.0032(x - 5)$$

$$CL_{5\%} = 0.0032x - 0.0061$$

For the rank order number of 5.0051, the interpolated 5% confidence limit value is:

$$CL_{5\%} = 0.0032(5.0051) - 0.0061$$

$$= 0.0099 \text{ or } 0.99\%$$

For the 95% confidence limit data above with:

$$(x_1, y_1) = (5, 0.0453) \text{ and}$$

$$(x_2, y_2) = (6, 0.0519)$$

the result is:

$$CL_{95\%} = 0.0453 + \frac{(0.0519 - 0.0453)}{(6 - 5)} * (x - 5)$$

$$CL_{95\%} = 0.0453 + 0.0066(x - 5)$$

$$CL_{95\%} = 0.0066x + 0.0123$$

For the rank order number of 5.0051, the interpolated 95% confidence limit value is:

$$CL_{95\%} = 0.0066(5.0051) + 0.0123$$

$$= 0.0453 \text{ or } 4.53\%$$

The previous table with all the 5% and 95% confidence limit values in place is provided below. The information which is no longer needed has been removed from the table.

**Table 5** - Sample Data Table with 5% and 95% Confidence Limit Values Supplied

Item	MR “Median Rank”	CL <sub>5%</sub> “5% confidence limits”	CL <sub>95%</sub> “95% confidence limits”	Loops	Time (hours)	Fail (F)/ Suspension (S)
A	0.0035	0.0003	0.0154	0	0	F
B	0.0085	0.0018	0.0236	0	0	F
C	0.0135	0.0041	0.0313	1	0.2	F
D	0.0185	0.0069	0.0385	2	0.8	F
E	-	-	-	3	1.0	S
F	0.0235	0.0099	0.0453	6	1.3	F
G	0.0285	0.0131	0.0520	11	2.1	F
H	0.0335	0.0166	0.0585	24	5.8	F
I	-	-	-	25	7.0	S
J	0.0386	0.0201	0.0649	40	8.9	F
K	0.0436	0.0237	0.0716	69	12.7	F

4.7 To smooth the data and determine if the sample data fits a Weibull distribution, it is desirable to transform the Weibull reliability function (and the sample data) so the transformed sample data will plot as a straight line. Since the Weibull reliability function is a double exponential, we will need to take the double logarithm of both sides of the function to “linearize” it.

From the original function:

$$R(t) = 1 - F(t) = \exp[-(t/\eta)^\beta]$$

since  $1 - F(t) = 1 - MR_i$ ,

$$1 - MR_i = \exp[-(t/\eta)^\beta]$$

from which we derive:

$$1/(1 - MR_i) = \exp(t/\eta)^\beta$$

$$\ln [1/ (1 - MR_i)] = (t/\eta)^\beta$$

$$\ln \ln [1/ (1 - MR_i)] = \beta \ln(t) - \beta \ln(\eta) \quad (20)$$

where:

$MR_i$  = the median rank values

$t$  = time

$\beta$  = shape parameter (to be determined from failure data)

$\eta$  = characteristic life (to be determined from failure data)

The median rank, the time and the 5% and 95% confidence limits will undergo the following transformation prior to plotting:

$$Y = \ln \ln [1/ (1 - MR_i)]$$

$$X = \ln(t)$$

$$Y_{5\%} = \ln \ln [1/ (1 - CL_{5\%})]$$

$$Y_{95\%} = \ln \ln [1/ (1 - CL_{95\%} )]$$

where:

$MR_i$  = the median rank values

$t$  = time to failure

$CL_{5\%}$  = the 5% confidence limit (not expressed as %)

$CL_{95\%}$  = the 95% confidence limit (not expressed as %)

The sample data with the data transformations supplied is shown below. The columns of the data table which are no longer used have been discarded. Also, the failure data for which time = 0, loops = 0 or were suspended will not be used on the Weibull plot and, therefore, will not be transformed.

**Table 6 - Sample Data Table with Data Transformations Supplied**

Item	MR “Median Rank”	Y	CL <sub>5%</sub>	Y <sub>5%</sub>	CL <sub>95%</sub>	Y <sub>95%</sub>	Time (hours)	X “ln(t)”	Loops
A	0.0035	-	0.0003	-	0.0154	-	0	-	0
B	0.0085	-	0.0018	-	0.0236	-	0	-	0
C	0.0135	-4.298	0.0041	-5.495	0.0313	-3.448	0.2	-1.61	1
D	0.0185	-3.981	0.0069	-4.973	0.0385	-3.238	0.8	-0.22	2
E	-	-	-	-	-	-	1.0	-	3
F	0.0235	-3.739	0.0099	-4.610	0.0453	-3.071	1.3	0.26	6
G	0.0285	-3.543	0.0131	-4.329	0.0520	-2.930	2.1	0.74	11
H	0.0335	-3.379	0.0166	-4.090	0.0585	-2.809	5.8	1.76	24
I	-	-	-	-	-	-	7.0	-	25
J	0.0386	-3.235	0.0201	-3.897	0.0649	-2.702	8.9	2.19	40
K	0.0436	-3.110	0.0237	-3.730	0.0716	-2.600	12.7	2.54	69

4.8 From the data points for the median ranks (with time > 0 and loops > 0), a line of best fit will be determined using linear regression (method of least squares). The linear regression line will have the familiar form:

$$Y = AX + B$$

The values for “A” and “B” will be determined using the following expressions:

$$A = \frac{n \sum X_i Y_i - \sum X_i \sum Y_i}{n \sum X_i^2 - (\sum X_i)^2} \quad (21)$$

$$B = \frac{\sum Y_i \sum X_i^2 - \sum X_i \sum X_i Y_i}{n \sum X_i^2 - (\sum X_i)^2} \quad (22)$$

where:

n = number of data points

X<sub>i</sub> = “X” value of data point where i goes from “1” to “n”

Y<sub>i</sub> = “Y” value of data point where i goes from “1” to “n”

**Table 7 - Data Points for Linear Regression**

data point, i =	“X <sub>i</sub> ” value	“Y <sub>i</sub> ” value
1	-1.61	-4.298
2	-0.22	-3.981
3	0.26	-3.739
4	0.74	-3.543
5	1.76	-3.379
6	2.19	-3.235
7	2.54	-3.110

**Table 8 - Summation Table for Linear Regression**

	X <sub>i</sub>	Y <sub>i</sub>	X <sub>i</sub> Y <sub>i</sub>	X <sub>i</sub> <sup>2</sup>
i = 1	-1.61	-4.298	6.9198	2.5921
i = 2	-0.22	-3.981	0.8758	0.0484
i = 3	0.26	-3.739	-0.9721	0.0676
i = 4	0.74	-3.543	-2.6218	0.5476
i = 5	1.76	-3.379	-5.9470	3.0976
i = 6	2.19	-3.235	-7.0847	4.7961
i = 7	2.54	-3.110	-7.8994	6.4516
Σ	5.66	-25.285	-16.7294	17.6010

$$A = \frac{(7 * -16.7294) - (5.66 * -25.285)}{(7 * 17.6010) - (5.66)^2}$$

$$= 0.285$$

$$B = \frac{(-25.285 * 17.6010) - (5.66 * -16.7294)}{(7 * 17.6010) - (5.66)^2}$$

$$= -3.843$$

The equation for the regression line, therefore, is:

$$Y_r = 0.285X - 3.843$$

Later when the regression line is used for plotting confidence limits, we will be determining new “X” values from known “Y” values. For this purpose, the regression line expression will also be solved for variable “X”.

$$X_r = (Y - B)/A$$

$$X_r = (Y + 3.843) / 0.285$$

4.9 Using the regression line expression, a new set of points will be added to the data table of values to plot. The “Y” values from table X (  $\ln \ln [1/ (1 - MR_i)]$  ) from the data table will be used to calculate another set of “X” values (labeled  $X_r$ ) will allow us to draw the regression line on the plot.

**Table 9** - New Data Points to Plot Regression Line

Item	Y	$X_r$
A	-	-
B	-	-
C	-4.298	-1.596
D	-3.981	-0.484
E	-	-
F	-3.739	0.365
G	-3.543	1.053
H	-3.379	1.628
I	-	-
I	-3.235	2.133
J	-3.110	2.572

**Table 10** - Data Table with only the values for plotting

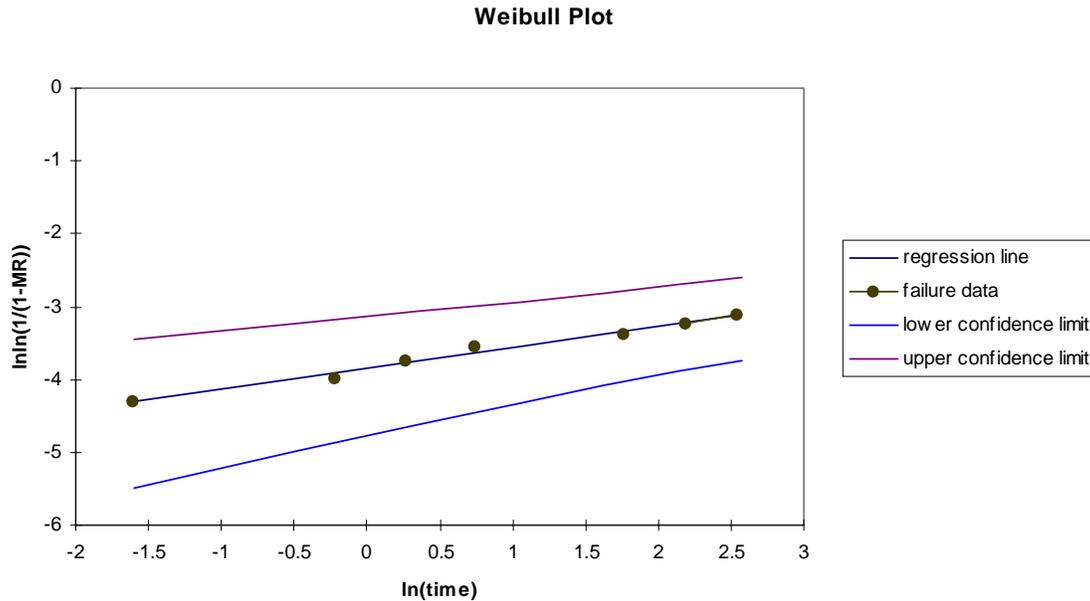
X	$X_r$	Y	$Y_{5\%}$	$Y_{95\%}$
-1.61	-1.596	-4.298	-5.495	-3.448
-0.22	-0.484	-3.981	-4.973	-3.238
0.26	0.365	-3.739	-4.610	-3.071
0.74	1.053	-3.543	-4.329	-2.930
1.76	1.628	-3.379	-4.090	-2.809
2.19	2.133	-3.235	-3.897	-2.702
2.54	2.572	-3.110	-3.730	-2.600

The Weibull Plot will consist of four items: a set of points, a regression line, an upper confidence limit curve and a lower confidence limit curve. The columns from Table 10 used to plot each item is given below:

**Table 11 - X and Y columns used for plotting**

	abscissa (X-axis)	ordinate (Y-axis)
Set of Points	X	Y
Regression Line	$X_r$	Y
Upper Confidence Limit	$X_r$	$Y_{95\%}$
Lower Confidence Limit	$X_r$	$Y_{5\%}$

The plot of the data in table 10 is shown below:



When the data points are closely clustered around the regression line, the behavior of the data can be described by the Weibull distribution. The confidence limits indicate our prediction accuracy based on the number of plotted data points. As more points are plotted, the confidence limits will move closer to the regression line. The confidence limits can be used in two ways.

The first use of the confidence limits is to determine the ranges for the expected percentage of failures for a specified time. To perform this exercise, one will draw a vertical line from the x-axis which intersects the specified “ $\ln(\text{time})$ ” and also intersects both confidence limit curves. Next, draw a horizontal line from each point of intersection with the confidence limit curves to the y-axis. Two values will be read from the y-axis. These values will need to be transformed back into median rank values using the following expression:

$$\text{rank \%} = 100 * ( 1 - 1 / \exp(\exp(y)) ) \quad (23)$$

where:

y = the value read from the y-axis where  
the horizontal line intersects  
rank % = cumulative failures

The two rank percentages, which we will call rank%<sub>low</sub> and rank%<sub>high</sub>, tell us that for a given screening time we are 90% confident that between rank%<sub>low</sub> and rank%<sub>high</sub> of the population will have failed.

The second use of the confidence limits is to determine the ranges for the expected time for a specified percentage of failures. To perform this exercise, one will draw a horizontal line from the y-axis which intersects the specified cumulative failure percent (transformed by  $\ln(\ln(1/(1-MR)))$ ) and also intersects both confidence limit curves. Next, draw a vertical line from each point of intersection with the confidence limit curves to the x-axis. Two values will be read from the x-axis. These values will need to be transformed back into time values using the following expression:

$$\text{time} = \exp(x) \quad (24)$$

where:

x = the value read from the x-axis where  
the vertical line intersects  
time = screening time

The two time values, which we will call time<sub>low</sub> and time<sub>high</sub>, tell us that for a given percentage of failures we are 90% confident that the failure percentage will be reached in the interval between time<sub>low</sub> and time<sub>high</sub>.

4.10 From the regression line expression, the values for  $\beta$  and  $\eta$  will be determined.

The regression line expression was:

$$Y_r = AX + B$$

Where:

$$A = 0.285$$

$$B = -3.843$$

The transformed Weibull function was:

$$\ln \ln [1 / (1 - MR)] = \beta \ln(t) - \beta \ln(\eta)$$

When compared with the regression line expression, we see that

$$Y_r = \ln \ln [1 / (1 - MR)]$$

$$A = \beta$$

$$X = \ln(t)$$

$$B = -\beta \ln(\eta)$$

Now that  $\beta$  has been found, we still need to find the value for  $\eta$  (which is the characteristic life). The characteristic life is the time at which  $F(t) = 0.632$  (the time when 63.2% of the population will have failed).

Since:  $R(t) = 1 - F(t)$

$$R(t) = 1 - 0.632$$

And:

$$1 - MR = R(t)$$

We get:

$$1 - MR = 0.368$$

Substituting into equation 20:

$$\ln \ln [ 1 / (1 - MR) ] = \beta \ln(t) - \beta \ln(\eta)$$

$$\ln \ln [ 1 / 0.368 ] = \beta \ln(t) - \beta \ln(\eta)$$

$$0 = \beta \ln(t) - \beta \ln(\eta)$$

Now solve for  $\eta$ :

$$\beta \ln(t) = \beta \ln(\eta)$$

$$\ln(t) = \ln(\eta)$$

$$\eta = t \quad (25)$$

using the regression line expression we get:

$$0 = A * X + B$$

$$X = -B/A$$

$$X = -[-3.843/0.285]$$

$$X = 13.484$$

Since:  $\ln(t) = X$

$$t = e^X$$

$$t = e^{13.484}$$

$$t = 717,989$$

$$\therefore \eta = 717,989$$

4.11 Now that  $\beta$  and  $\eta$  have been determined, their values will be substituted into the Weibull hazard rate function. The Weibull hazard rate function will also be used to generate a set of points to plot the Weibull hazard function.

$$h(t) = (\beta/\eta) * (t/\eta)^{(\beta-1)}$$

$$h(t) = (0.285/717,989) * (t/717,989)^{(0.285-1)}$$

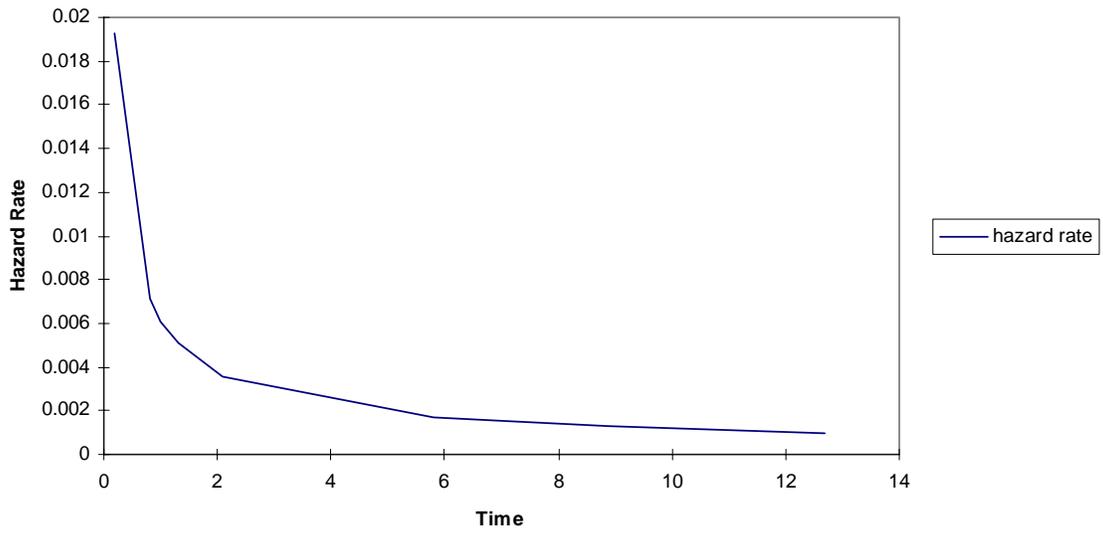
$$h(t) = (0.285/717,989) * (t/717,989)^{(-0.715)}$$

The hazard rate plot will have time on the abscissa (X-axis) and  $h(t)$ , the instantaneous hazard rate, on the ordinate (Y-axis). The hazard rate expression will not be evaluated for time values of zero.

**Table 12** - Data Points for Weibull Hazard Rate Plot

Item	Time (hours)	$h(t)$
A	-	-
B	-	-
C	0.2	0.019303
D	0.8	0.007164
E	1.0	0.006107
F	1.3	0.005063
G	2.1	0.003593
H	5.8	0.001738
I	7.0	0.001519
J	8.9	0.001279
K	12.7	0.000992

**Hazard Rate Plot**



4.12 Now the Weibull hazard rate function can be used to determine the screening duration of an screening process. One method is to use the products known MTBF value. The MTBF value is essentially the inverse of the hazard rate value that is modeled by the Weibull function when the shape parameter is one. The screening time termination is selected when the decreasing exponential of the hazard rate function (shape parameter < 1) crosses a horizontal line which signifies the 1/MTBF value. This method can lead to unreasonably long screening times.

Another method selects the screening time termination at the moment that the hazard rate curve slope value approaches a small negative number. A slope value that is a small negative number suggests that the hazard rate is nearly constant. The smaller the negative number the longer the screening time. Slope values less than or equal to - 0.00005 are generally used.

To find the required screening time for a particular hazard rate, the hazard rate expression will be differentiated and solved for time.

hazard rate expression:

$$h(t) = (\beta/\eta) * (t/\eta)^{(\beta-1)}$$

the differentiated hazard rate expression:

$$h'(t) = [\beta * (\beta - 1) * t^{(\beta-2)}] / \eta^\beta \quad (26)$$

the differentiated hazard rate expression solved for t (time):

$$t = [(\eta^\beta * h'(t)) / (\beta * (\beta - 1))]^{1/(\beta-2)} \quad (27)$$

where:

$\eta$  = a constant (characteristic life), determined from failure data

$\beta$  = a constant (shape parameter), determined from failure data

$h'(t)$  = hazard rate slope (a small negative number)

t = screening time

for the sample data, the optimum screening time is:

$$t = [(717,989^{0.285} * -0.00005) / (0.285 * (0.285 - 1))]^{1/(0.285-2)}$$

$$t = 13.55 \text{ hours}$$

4.13 A more complicated and more accurate approach that is used to determine screening duration is with Bayesian Analysis. Using Bayesian Analysis, the probabilities of a failure belonging to the weak subpopulation and to the main population are evaluated to find the cumulative failure percentage at which the weak subpopulation has been eliminated. This failure percentage value will be transformed and substituted into the regression formula to find the proper screening time.

The parameters for the weak subpopulation is:

$$\beta_1 \text{ and } \eta_1$$

where:

- $\beta_1$  = the shape parameter from the Weibull Analysis
- $\eta_1$  = the characteristic life from the Weibull Analysis

The parameters for the main population is:

$$\beta_2 \text{ and } \eta_2$$

where:

- $\beta_2$  = the shape parameter for constant hazard rate (=1)
- $\eta_2$  = the MTTF of the product (we will use 100,000hrs.)

The probability of a failure “i” belonging to the weak subpopulation is:

$$P_i := \frac{\frac{f_1}{t_i}}{\frac{f_1}{t_i} + \frac{f_2}{t_i}} \quad (29)$$

where:

$$\frac{f_1}{t_i} := \left(\frac{\beta_1}{\eta_1}\right) \cdot \exp\left[-\left(\frac{t_i}{\eta_1}\right)^{\beta_1}\right] \cdot \left(\frac{t_i}{\eta_1}\right)^{(\beta_1-1)} \quad (30)$$

and

$$\frac{f_2}{t_i} := \left(\frac{\beta_2}{\eta_2}\right) \cdot \exp\left[-\left(\frac{t_i}{\eta_2}\right)^{\beta_2}\right] \cdot \left(\frac{t_i}{\eta_2}\right)^{(\beta_2-1)} \quad (31)$$

$t_i$  = the time to failure for a member of the failing population

If expressions 29,30 and 31 are combined, we get:

$$P_i := \frac{\left(\frac{\beta_1}{\eta_1}\right) \cdot \exp\left[-\left(\frac{t_i}{\eta_1}\right)^{\beta_1}\right] \cdot \left(\frac{t_i}{\eta_1}\right)^{(\beta_1-1)}}{\left[\left(\frac{\beta_1}{\eta_1}\right) \cdot \exp\left[-\left(\frac{t_i}{\eta_1}\right)^{\beta_1}\right] \cdot \left(\frac{t_i}{\eta_1}\right)^{(\beta_1-1)}\right] + \left[\left(\frac{\beta_2}{\eta_2}\right) \cdot \exp\left[-\left(\frac{t_i}{\eta_2}\right)^{\beta_2}\right] \cdot \left(\frac{t_i}{\eta_2}\right)^{(\beta_2-1)}\right]} \quad (32)$$

The proportion of the weak subpopulation is given by:

$$\bar{p} := \frac{\sum_i P_i}{N} \quad (33)$$

If expression 32 and 33 are combined, we get:

$$\bar{p} := \frac{\sum_i \left[ \frac{\left(\frac{\beta_1}{\eta_1}\right) \cdot \exp\left[-\left(\frac{t_i}{\eta_1}\right)^{\beta_1}\right] \cdot \left(\frac{t_i}{\eta_1}\right)^{(\beta_1-1)}}{\left[\left(\frac{\beta_1}{\eta_1}\right) \cdot \exp\left[-\left(\frac{t_i}{\eta_1}\right)^{\beta_1}\right] \cdot \left(\frac{t_i}{\eta_1}\right)^{(\beta_1-1)}\right] + \left[\left(\frac{\beta_2}{\eta_2}\right) \cdot \exp\left[-\left(\frac{t_i}{\eta_2}\right)^{\beta_2}\right] \cdot \left(\frac{t_i}{\eta_2}\right)^{(\beta_2-1)}\right]} \right]}{N} \quad (34)$$

Where:

- N = is the sample size (pass, fail and suspended units)
- p-bar = the proportion of the sample belonging to the weak subpopulation
- $\beta_1$  = the shape parameter from the Weibull Analysis
- $\eta_1$  = the characteristic life from the Weibull Analysis
- $\beta_2$  = the shape parameter for constant hazard rate (=1)
- $\eta_2$  = the MTTF of the product (we will use 100,000hrs.)
- $t_i$  = the time to failure for a member of the failing population

The values for time,  $f_1/t_i$ ,  $f_2/t_i$  and  $P_i$  are given in the table 13 below:

**Table 13** - Intermediate values for Bayesian Analysis

Failure Times	$f_1/t_i$	$f_2/t_i$	$P_i$
0	0	0	1
0	0	0	1
0.2	0.019043	0.000010	0.999475
0.8	0.007021	0.000010	0.998578
1.3	0.004947	0.000010	0.997983
2.1	0.003499	0.000010	0.997150
5.8	0.001677	0.000010	0.994072
8.9	0.001229	0.000010	0.991929
12.7	0.000949	0.000010	0.989572

Note that the  $P_i$  values for failures at  $t_i = 0$  are assigned a probability of one since we are 100% certain that they do not belong to the main population and the suspended units have been excluded from table 13 since they were not failed units.

When we sum the last column of table 13, we get:

$$\sum P_i = 8.968759$$

$$p\text{-bar} = \sum P_i/N$$

$$= 8.968759/200$$

$$= 0.044844 \text{ or } 4.5\%$$

This number implies that all the members of the weak subpopulation will have failed when 4.5% of the sample has failed. The p-bar value will be transformed as was done with the median rank values to allow it to be used with the regression formula. Using equation 20 and the previously determined values for  $\beta$  and  $\eta$

$$\ln \ln [1/ (1 - MR_i)] = \beta \ln(t) - \beta \ln(\eta)$$

$$\ln \ln [1/(1 - 0.045)] = 0.285\ln(t) - 0.285\ln(717,989)$$

$$-3.078 = 0.285\ln(t) - 3.843$$

$$\ln(t) = 2.68$$

$$\mathbf{t \text{ (screening time) = 14.6 hours}}$$

## 5.0 Summary

The Weibull function can effectively model failure behavior of products during their normal mission mode or during accelerated environments. Actual product failure behavior is used to empirically determine the two parameters of a two parameter Weibull function. These parameters are the shape parameter and the characteristic life. For products that have a minimum guaranteed life before the first failure occurs or can be detected should use the three parameter Weibull model. The additional parameter of the three parameter Weibull model over the two parameter Weibull model is called the minimum life.

The method used in this paper to determine the Weibull parameters is called the probability plotting technique. This technique is the best technique to demonstrate the versatility and capability of the Weibull function to new users. Once the Weibull parameters have been determined, the Weibull hazard rate function can be used in one of several ways to determine the optimum screening duration.

A program which implements the functions described in this paper which can be used to determine environmental stress screening durations is available from Accolade Engineering Solutions. We may be reached at [info@accoladeeng.com](mailto:info@accoladeeng.com).